ISRAEL JOURNAL OF MATHEMATICS 108 (1998), 1–12

MINIMAL RATES OF ENTROPY CONVERGENCE FOR COMPLETELY ERGODIC SYSTEMS

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ABSTRACT

If (X,T) is a completely ergodic system, then there exists a positive monotone increasing sequence $\{a_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} a_n = \infty$ and a positive concave function g defined on $[1,\infty)$ for which $g(x)/x^2$ is *not* integrable such that

$$\liminf_{n\to\infty} H(\alpha_0^{n-1})/a_n > 0 \quad \text{and} \quad \limsup_{n\to\infty} H(\alpha_0^{n-1})/g(\log_2 n) > 0$$

for all nontrivial partitions α of X into two sets.

1. Introduction

In this paper we continue our investigation of minimal entropy convergence rates that was begun in [3]. Let us review some standard notation: If $(X,\mathcal{B},\mu)$ is a probability space, $T : X \to X$ a measure-preserving transformation and α a finite partition of X, then the **n-th refinement of** α under T is denoted by α_0^{n-1}. Furthermore, if $f(x) := -x\log_2 x$ for $x \in [0,1]$, then the **entropy of** α is defined as

$$H(\alpha) := \sum_{A\in\alpha} f(\mu(A)).$$

A measure-preserving transformation T is called completely ergodic if T^n is ergodic for all $n \in \mathbb{Z}$. The following general result was shown in [3].

Received April 6, 1997

1.1 THEOREM: *Let (X,T) be a completely ergodic system and assume that g is a positive monotone increasing function defined on $[1,\infty)$ which satisfies the condition $\int_1^\infty g(x)/x^2\,dx < \infty$. If α is a partition of X into two sets of positive measure, then*

$$\limsup_{n\to\infty} \frac{H\left(\alpha_0^{n-1}\right)}{g(\log_2 n)} = \infty.$$

Equivalently, if $\{X_n \mid n \in \mathbb{Z}\}$ is the completely ergodic stationary process with binary state space corresponding to the refinements of α under T, then

$$\limsup_{n\to\infty} \frac{H(X_1,\ldots,X_n)}{g(\log_2 n)} = \infty.$$

A typical choice for $g(x)$ is x^α for some $\alpha \in (0,1)$, but also stronger rates like, for example, $x/(\log_2 x)^{1+\alpha}$ do still satisfy the assumption $\int_1^\infty g(x)/x^2\,dx < \infty$. We wish to emphasize that the concept of entropy convergence rates is mainly of interest for the case of zero entropy systems. In fact, the statement above is trivially true for systems with strictly positive entropy (K-automorphisms).

In the language of information theory, Theorem 1.1 says that the average information conveyed by the process has to grow in the limit superior at the given rate $g(\log_2 n)$. From the point of view of ergodic theory, the theorem gives us a universal lower bound for entropy convergence rates of measure-preserving systems in the limit superior. (Improvements of this result for rank one and rank one mixing transformations were given in [2], [3] and [4].) A natural question is whether a similar result can be established for entropy convergence rates in the limit inferior. The purpose of the present paper is to show that this is indeed the case for completely ergodic systems, although the lower bound for rates in the limit inferior is not universal (see Theorem 4.1). For each given completely ergodic system there is a lower bound for entropy convergence rates in the limit inferior, but depending on the system the rate may be arbitrarily slow (as can be demonstrated easily by using the methods developed in [2], Chapter 4). A possible question for further research concerns the size of the set of all zero entropy systems for which there exist partitions generating entropy rates in the limit inferior slower than a given subexponential rate. It might, for example, be possible to obtain a result similar in nature to the theorem which states that the set of weakly mixing measure-preserving systems is residual and the set of mixing measure-preserving systems is of first category (see [5], p. 71).

In [2] we showed that rates in the limit inferior and limit superior can actually be used to distinguish systems up to isomorphism by constructing certain classes of examples. However, it is conceivable that for 'most' systems we will not be able to find a lower bound for entropy convergence rates in the limit superior that is different from the general lower bound given by Theorem 1.1 and consequently we would not be able to use rates in the limit superior to distinguish those systems up to isomorphism. In Section 5 we will show that this is not the case if we assume the systems to be completely ergodic. In other words, for completely ergodic systems the concept of entropy convergence rates in the limit superior is always a non-trivial one.

We wish to mention that in [7] and [8] Shields discussed problems similar to those presented in this paper and he also used cutting and stacking methods in [6] that closely resemble our approach to the proof of Theorem 5.3.

2. Periods of 01-names

In this section we will mainly review some of the definitions and propositions that we already discussed in [3]. The definitions in this paper are actually slightly modified compared to [3], because we will work exclusively with 01-names of length 2^n for some $n \in \mathbb{N}$. Throughout this section we will assume that we are given a measure-preserving system (X, T). The union of all levels in a Rokhlin tower τ will be denoted by $|\tau|$.

2.1 Definition: If α is a finite partition of X and $F \in \mathcal{B}$, then we define the **entropy of α restricted to F** as

$$H_F(\alpha) := \sum_{A\in\alpha} f(\mu(A \cap F)).$$

2.2 LEMMA: *If α is a finite partition of X and $F, G \in \mathcal{B}$ with $G \subset F$, then*

$$H_F(\alpha) > H_G(\alpha) - 2.$$

2.3 Definition: For any $E \in \mathcal{B}$, $x \in X$, $n \in \mathbb{N}$ and $s \in \{0,1\}^{2^n}$ we define

$$\begin{aligned} &\alpha(E) := \{E, X \setminus E\}, \\ &s_n^E(x) := (\chi_E(x), \chi_E(Tx), \ldots, \chi_E(T^{2^n-1}x)) \quad \text{and} \\ &A_n^E(s) := (s_n^E)^{-1}(\{s\}). \end{aligned}$$

With these definitions it is easy to see that $\alpha(E)_0^{2^n-1} = \{A_n^E(s) | \ s \in \{0,1\}^{2^n}\}$. In order to understand how the measure of $A_n^E(s)$ depends on the properties of s, we need to define the period of a 01-name.

2.4 Definition: For $s \in \{0,1\}^{2^n}$ we define the **period** of s as

$$p_n(s) := \min\{k \in \{1,\ldots,2^n-1\} \mid s_i = s_{i+k} \text{ for all } i \in \{0,\ldots,2^n-k-1\}\} \cup \{2^n\}.$$

The proofs of the following two lemmas are very simple and will be omitted (cf. [3], Lemma 1.6 and Lemma 1.8).

2.5 Lemma: *If $E \in \mathcal{B}$ and $s \in \{0,1\}^{2^n}$, then*

$$\mu(A_n^E(s)) \leq \frac{1}{p_n(s)}.$$

2.6 Lemma: *Let α be a finite partition of X, $F \in \mathcal{B}$ and $c > 0$ such that*

$$\mu(A \cap F) \leq c \quad \text{for all } A \in \alpha.$$

Then

$$H_F(\alpha) \geq \mu(F) \log_2 \frac{1}{c}.$$

2.7 Definition: Let $s \in \{0,1\}^{2^n}$, $E \in \mathcal{B}$ and $1 \leq p \leq q \leq 2^n$. Then we define

$$P_p^q(n,E) := \{x \in X \mid p \leq p_n(s_n^E(x)) \leq q\}.$$

3. The main lemma

In this section we will show that in every completely ergodic system the measure of the sets $P_1^{2^n}(m,E)$ decreases with n and m at a rate that in a certain sense is independent of E. More precisely, if n is sufficiently large and if m is sufficiently large compared to n, then the measure of $P_1^{2^n}(m,E)$ will be small and the speed at which we have to increase m relative to n will eventually not depend on E. This will allow us to find non-trivial common lower bounds for the entropy convergence rates generated by the partitions $\alpha(E)$.

It is well known that every aperiodic transformation is isomorphic to an interval exchange transformation on the interval $[0,1]$ (see [1]). Since every completely ergodic system is in particular aperiodic, we may assume our probability space to be $([0,1],\mathcal{B},\mu)$ with μ being Lebesgue measure and $\mathcal{B}$ being the σ-algebra of all Lebesgue measurable subsets of $[0,1]$.

3.1 Lemma: *If $([0,1],T)$ is completely ergodic and $\varepsilon > 0$, then there exists a strictly increasing sequence $\{N_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that for all $E \in \mathcal{B}$ with $0 < \mu(E) < 1$ we can find a $K \in \mathbb{N}$ such that*

$$\mu(P_1^{2^n}(N_n,E)) < \varepsilon \quad \text{for all } n \geq K.$$

Proof: For $n \in \mathbb{N}$ we define

$$\beta_n := \left\{ \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \mid k \in \{1, \ldots, 2^n\} \right\}.$$

For a fixed $n \in \mathbb{N}$ we can use the ergodic theorem and the complete ergodicity of T to find an $M_n \in \mathbb{N}$ and a set $S_n \in \mathcal{B}$ such that for all $m \geq M_n$, $x \in S_n$ and $I \in \beta_n$ we have

$$\left| \frac{1}{m} \sum_{k=0}^{m-1} \chi_I((T^{2^n!})^k(x)) - \mu(I) \right| < \frac{1}{n2^n} \tag{1}$$

and

$$\mu(S_n) > 1 - \frac{1}{n}. \tag{2}$$

Now we define $N_n := [\log_2(M_n(2^n!))] + 1$. Let $\varepsilon > 0$ and $E \in \mathcal{B}$ such that $0 < \mu(E) < 1$ and let $\eta := \min\{\varepsilon/2, (1-\mu(E))/3, \mu(E)/3\}$. Then for all $n \in \mathbb{N}$ we define

$$F_n := \bigcup \{I \in \beta_n \mid 2^n \mu(I \cap E) > 1/2\}.$$

According to the Lebesgue density theorem we can find a $K > 1/\eta$ such that for all $n \geq K$ we have

$$\mu(E \triangle F_n) < \eta^2. \tag{3}$$

Furthermore, using inequality (1) we obtain

$$\left| \frac{1}{m} \sum_{k=0}^{m-1} \chi_{F_n}((T^{2^n!})^k(x)) - \mu(F_n) \right| < \frac{1}{n} \quad \text{for all } m \geq M_n \text{ and } x \in S_n. \tag{4}$$

For $n \in \mathbb{N}$ we define

$$R_n := \left\{ x \in [0,1] \;\middle|\; \frac{1}{M_n} \sum_{k=0}^{M_n-1} \left| \chi_E((T^{2^n!})^k(x)) - \chi_{F_n}((T^{2^n!})^k(x)) \right| > \eta \right\}.$$

Then we have

$$\begin{aligned} \eta\,\mu(R_n) &\leq \int_0^1 \frac{1}{M_n} \sum_{k=0}^{M_n-1} \left| \chi_E((T^{2^n!})^k(x)) - \chi_{F_n}((T^{2^n!})^k(x)) \right| d\mu \\ &= \frac{1}{M_n} \sum_{k=0}^{M_n-1} \int_0^1 |\chi_E(x) - \chi_{F_n}(x)|\, d\mu \\ &= \mu(E \triangle F_n). \end{aligned}$$

Therefore, for all $n \geq K$ we have

$$\mu(R_n) < \eta \quad \text{(by (3))}. \tag{5}$$

Now let $x \in S_n \setminus R_n$ and $n \geq K$. Then

$$\begin{aligned}\left|\frac{1}{M_n}\sum_{k=0}^{M_n-1}\chi_E((T^{2^n!})^k(x)) - \mu(E)\right| &\leq \left|\frac{1}{M_n}\sum_{k=0}^{M_n-1}\chi_{F_n}((T^{2^n!})^k(x)) - \mu(F_n)\right| \\ &+ \frac{1}{M_n}\sum_{k=0}^{M_n-1}\left|\chi_E((T^{2^n!})^k(x)) - \chi_{F_n}((T^{2^n!})^k(x))\right| + |\mu(F_n) - \mu(E)| \\ &< \frac{1}{n} + 2\eta < 3\eta \quad \text{(by (3), (4), the definition of } R_n \text{ and the choice of } K).\end{aligned} \tag{6}$$

If $x \in P_1^{2^n}(N_n, E)$, then $p_{N_n}(x)$ divides $2^n!$ and, since $M_n(2^n!) < 2^{N_n}$, it is easy to see that

$$\frac{1}{M_n}\sum_{k=0}^{M_n-1}\chi_E((T^{2^n!})^k(x)) = \begin{cases} 0, & \text{if } x \in [0,1] \setminus E, \\ 1, & \text{if } x \in E. \end{cases}$$

Hence

$$\left|\frac{1}{M_n}\sum_{k=0}^{M_n-1}\chi_E((T^{2^n!})^k(x)) - \mu(E)\right| \geq \min\{\mu(E), 1 - \mu(E)\} \geq 3\eta.$$

Using (6) we can therefore conclude that for all $n \geq K$ we have $P_1^{2^n}(N_n, E) \subset [0,1] \setminus (S_n \setminus R_n)$ and this shows that

$$\mu(P_1^{2^n}(N_n, E)) \leq 1 - \mu(S_n \setminus R_n) < 2\eta \leq \varepsilon \quad \text{(by (2), (5) and the choice of } K).$$

Since we can clearly choose the sequence $\{N_n\}_{n=1}^{\infty}$ to be strictly increasing, the proof is complete. ∎

4. Convergence rates in the limit inferior

We will show that for every completely ergodic system there exists a non-trivial lower bound for entropy convergence rates in the limit inferior.

4.1 THEOREM: *If $([0,1],T)$ is completely ergodic, then there exists a positive monotone increasing sequence $\{a_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} a_k = \infty$ such that for all $E \in \mathcal{B}$ with $0 < \mu(E) < 1$ we have*

$$\liminf_{k\to\infty} \frac{H(\alpha(E)_0^{k-1})}{a_k} \geq 1.$$

Equivalently, if $\{X_n | \, n \in \mathbb{Z}\}$ is the completely ergodic stationary process with binary state space corresponding to the refinements of $\alpha(E)$ under T, then

$$\liminf_{k\to\infty} \frac{H(X_1,\dots,X_k)}{a_k} \geq 1.$$

Proof: First we apply Lemma 3.1 with $\varepsilon := 1/2$ to find a sequence $\{N_n\}_{n=1}^{\infty}$ as described in the statement of Lemma 3.1. Then we define the sequence $\{a_k\}_{k=1}^{\infty}$ as follows: If $1 \leq k < 2^{N_1}$ then $a_k := 1$, and if $k \in \{2^{N_n},\dots,2^{N_{n+1}}-1\}$ then $a_k := n$. This definition implies clearly that $\{a_k\}_{k=1}^{\infty}$ is a monotone increasing sequence with $\lim_{k\to\infty} a_k = \infty$. Let $E \in \mathcal{B}$ such that $0 < \mu(E) < 1$. If $s \in \{0,1\}^{2^{N_n}}$ with $p_{N_n}(s) > 2^n$, then

$$\mu(A_{N_n}^E(s)) < \frac{1}{2^n} \quad \text{(by Lemma 2.5).}$$

Therefore, $\alpha(E)_0^{2^{N_n}-1}$ induces a partition on $P_n := P_{2^n+1}^{2^{N_n}}(N_n,E)$ such that

$$\mu(A \cap P_n) < \frac{1}{2^n} \quad \text{for all } A \in \alpha(E)_0^{2^{N_n}-1}.$$

Hence

$$\begin{aligned} H(\alpha(E)_0^{2^{N_n}-1}) &\geq H_{P_n}(\alpha(E)_0^{2^{N_n}-1}) - 2 \quad \text{(by Lemma 2.2)} \\ &\geq \mu(P_n)n - 2 \quad \text{(by Lemma 2.6).} \end{aligned}$$

Using the definition of $\{a_k\}_{k=1}^{\infty}$ we obtain for all $k \in \{2^{N_n},\dots,2^{N_{n+1}}-1\}$

$$\frac{H(\alpha(E)_0^{k-1})}{a_k} \geq \frac{H(\alpha(E)_0^{2^{N_n}-1})}{n} \geq \mu(P_n) - \frac{2}{n}.$$

Since Lemma 3.1 implies that $\liminf_{n\to\infty} \mu(P_n) \geq 1/2$, we conclude that

$$\liminf_{k\to\infty} \frac{H(\alpha(E)_0^{k-1})}{a_k} \geq \liminf_{n\to\infty} \frac{\mu(P_n)n-2}{n} \geq \frac{1}{2}.$$

Replacing a_k by $a_k/2$ gives us the desired conclusion. ∎

4.2 Remark: The statement of Theorem 4.1 is obviously no longer true if we drop the assumption of complete ergodicity, because the system $([0,1],T)$ could then have a periodic factor. But even if we restrict the class of partitions to those for which the maximum measure of the atoms in α_0^{n-1} converges to 0, we will in general not be able to find a universal lower bound for entropy convergence rates in the limit inferior (see Proposition 3.12 in [3]).

5. Convergence rates in the limit superior

In this final section we will prove that for every completely ergodic system we can find a lower bound for entropy convergence rates in the limit superior, which is non-trivial with respect to the general statement of Theorem 1.1. First we need some more notation.

5.1 Definition: Let $\tau = (B, TB, \ldots, T^{2^n-1}B)$ be a Rokhlin tower, $E \in \mathcal{B}$, $0 \le k \le n$ and $0 \le i \le k$. Then we define

$$B_k := \bigcup_{j=0}^{2^{n-k}-1} T^{j2^k} B \quad \text{and} \quad x_{\tau,k}^E(i) := \frac{\mu(P_{2^{i-1}+1}^{2^i}(k,E) \cap B_k)}{\mu(B_k)}.$$

The proof of the following lemma will be omitted, since it is almost identical to the proof of Lemma 5.2 in [4]. (*Note*: If τ is a Rokhlin tower, then we denote by $|\tau|$ the union of all the levels of τ.)

5.2 LEMMA: *If $\tau = (B, TB, \ldots, T^{2^n-1}B)$ is a Rokhlin tower, $E \in \mathcal{B}$ and $3 \le k \le n$ then*

$$H(\alpha(E)_0^{2^{k-1}-1}) \ge \frac{\mu(|\tau|)}{4} \sum_{i=0}^{k} i x_{\tau,k}^E(i) - 7 - \log_2 k.$$

5.3 THEOREM: *If $([0,1],T)$ is completely ergodic, then there exists a positive concave function g defined on $[1,\infty)$ with $\int_1^\infty g(x)/x^2 dx = \infty$, such that for all $E \in \mathcal{B}$ with $0 < \mu(E) < 1$ we have*

$$\limsup_{n\to\infty} \frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} \ge 1.$$

Equivalently, if $\{X_n |\ n \in \mathbb{Z}\}$ is the completely ergodic stationary process with binary state space corresponding to the refinements of $\alpha(E)$ under T, then

$$\limsup_{n\to\infty} \frac{H(X_1, \ldots, X_n)}{g(\log_2 n)} \ge 1.$$

Proof: We choose a sequence $\{N_n\}_{n=1}^{\infty}$ as explained in Lemma 3.1 for $\varepsilon := 1/8$. Then we choose a subsequence $\{N_{n_k}\}_{k=1}^{\infty}$ such that for all $k > 1$ we have

$$n_k > N_{n_{k-1}} \tag{1}$$

and

$$\frac{N_{n_{k+1}}}{N_{n_k}} > 2\frac{N_{n_k}}{N_{n_{k-1}}} > 2e. \tag{2}$$

In order to simplify our notation we set $m_k := N_{n_k}$. Now we define a function $h : [1, \infty) \to [1, \infty)$ recursively as follows:

$$\begin{aligned} &h(x) := x \quad \text{for } x \in [1, m_1] \quad \text{and} \\ &h(x) := h(m_k) + \frac{x - m_k}{\ln(m_{k+1}) - \ln(m_k)} \quad \text{for } x \in (m_k, m_{k+1}]. \end{aligned}$$

It is easy to see that h is continuous and has constant slope $(\ln(m_{k+1}/m_k))^{-1} < 1$ (by (2)) along each interval (m_k, m_{k+1}). Thus, (2) implies that h is concave and therefore

$$g(x) := h(x) + \ln^2(x + 2)$$

is a positive concave function on $[1, \infty)$. From the concavity of g it follows that $g(x)/x$ is monotone decreasing. Furthermore, (2) implies that

$$\lim_{\substack{x\to\infty \\ x\neq m_k}} h'(x) = \lim_{k\to\infty} \frac{1}{\ln(m_{k+1}/m_k)} \leq \lim_{k\to\infty} \frac{1}{(k-1)\ln 2 + \ln(m_2/m_1)} = 0.$$

Hence

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{x\to\infty} \frac{h(x)}{x} = 0. \tag{3}$$

For $k > 1$ we have

$$\int_{m_{k-1}}^{m_k} \frac{h(x)}{x^2}dx = -\frac{h(x)}{x}\bigg|_{m_{k-1}}^{m_k} + \int_{m_{k-1}}^{m_k} \frac{h'(x)}{x}dx = -\frac{h(x)}{x}\bigg|_{m_{k-1}}^{m_k} + 1. \tag{4}$$

This shows that $\int_1^{\infty} g(x)/x^2 dx = \infty$. Now let $E \in \mathcal{B}$ with $0 < \mu(E) < 1$. According to Lemma 3.1 and (3) we can find a $K \geq 3$ such that for all $k \geq K$ we have

$$\mu(P_1^{2^{n_k}}(m_k, E)) < \frac{1}{8} \tag{5}$$

and

$$\text{(6)} \qquad \frac{g(m_k)}{m_k+2} + \frac{g(m_k+1)}{(m_k+1)^2} - \frac{h(x)}{x}\bigg|_{m_{k-1}}^{m_k} + \int_{m_{k-1}}^{m_k} \frac{\ln^2(x+e)}{x^2}dx < 1.$$

Now let $k \geq K$ and let $\tau = (B, TB, \ldots, T^{2^{m_k+1}-1}B)$ be a Rokhlin tower with

$$\text{(7)} \qquad \mu(|\tau|) > \frac{1}{2}.$$

For all $x \in [0,1]$ and all $j \in \{0, \ldots, 2^{m_k} - 1\}$ the 01-name $s^E_{m_k}(T^j x)$ is a subname of $s^E_{m_k+1}(x)$ and therefore

$$p_{m_k}(s^E_{m_k}(T^j x)) \leq p_{m_k+1}(s^E_{m_k+1}(x)).$$

Hence

$$\bigcup_{j=0}^{2^{m_k}-1} T^j(B \cap P_1^{2^{n_k}}(m_k+1, E)) \subset P_1^{2^{n_k}}(m_k, E).$$

Therefore we can use (5) and (7) to conclude that

$$\frac{\mu(B \cap P_1^{2^{n_k}}(m_k+1, E))}{\mu(B)} < \frac{1}{2}.$$

Now let $x_i := x^E_{\tau, m_k+1}(i)$ for all $i \in \{0, \ldots, m_k+1\}$. Then the previous inequality shows that

$$\text{(8)} \qquad \sum_{i=n_k+1}^{m_k+1} x_i > \frac{1}{2}.$$

Furthermore, it is easy to see that for all $j \in \{0, \ldots, m_k+1\}$ we have

$$x_i \leq x^E_{\tau,j}(i) \quad \text{for all } i \in \{0, \ldots, j\}.$$

Therefore Lemma 5.2 and (7) imply that

$$\text{(9)} \qquad H(\alpha(E)_0^{2^{j-1}-1}) \geq \frac{1}{8}\sum_{i=0}^{j} i x_i - 7 - \log_2 j \quad \text{for all } j \in \{0, \ldots, m_k+1\}.$$

Now we wish to show that

$$\text{(10)} \qquad \max_{j \in \{n_k+1, \ldots, m_k+1\}} \frac{1}{g(j-1)} \sum_{i=n_k+1}^{j} i x_i > \frac{1}{4}.$$

If not, then we have

$$4\sum_{i=n_k+1}^{j} ix_i \le g(j-1) \quad \text{for all } j\in\{n_k+1,\dots,m_k+1\} \tag{11}$$

and we obtain

$$\begin{aligned}\sum_{j=n_k+1}^{m_k+1}\frac{g(j-1)}{j(j+1)} &\ge 4\sum_{j=n_k+1}^{m_k+1}\frac{\sum_{i=n_k+1}^{j} ix_i}{j(j+1)} \qquad \text{(by (11))}\\ &= 4\sum_{i=n_k+1}^{m_k+1} ix_i \sum_{j=i}^{m_k+1}\frac{1}{j(j+1)}\\ &= 4\sum_{i=n_k+1}^{m_k+1} ix_i\left(\frac{1}{i}-\frac{1}{m_k+2}\right)\\ &> 2-\frac{g(m_k)}{m_k+2} \qquad \text{(by (8) and (11)).}\end{aligned} \tag{12}$$

Furthermore,

$$\begin{aligned}\sum_{j=n_k+1}^{m_k+1}\frac{g(j-1)}{j(j+1)} &< \sum_{j=n_k+1}^{m_k+1}\frac{g(j)}{j^2} \qquad (g \text{ is increasing})\\ &< \int_{n_k}^{m_k}\frac{g(x)}{x^2}dx+\frac{g(m_k+1)}{(m_k+1)^2} \qquad (g(x)/x \text{ is decreasing})\\ &< \int_{m_{k-1}}^{m_k}\frac{g(x)}{x^2}dx+\frac{g(m_k+1)}{(m_k+1)^2} \qquad \text{(by (1))}\\ &= \int_{m_{k-1}}^{m_k}\frac{\ln^2(x+2)}{x^2}dx+\frac{g(m_k+1)}{(m_k+1)^2}-\frac{h(x)}{x}\Big|_{m_{k-1}}^{m_k}+1 \qquad \text{(by (4)).}\end{aligned} \tag{13}$$

Using (6), (12) and (13) we obtain the contradiction $2>2$ and this proves (10). Now we apply (9) and (10) to conclude that

$$\max_{j\in\{n_k+1,\dots,m_k+1\}}\frac{H(\alpha(E)_0^{2^{j-1}-1})}{g(j-1)} > \frac{1}{32}-\max_{j\in\{n_k+1,\dots,m_k+1\}}\frac{7+\log_2 j}{\ln^2(j+1)}.$$

Finally, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} &\ge \limsup_{n\to\infty}\frac{H(\alpha(E)_0^{2^{n-1}-1})}{g(n-1)}\\ &\ge \limsup_{k\to\infty}\max_{j\in\{n_k+1,\dots,m_k+1\}}\frac{H(\alpha(E)_0^{2^{j-1}-1})}{g(j-1)}\\ &\ge \limsup_{k\to\infty}\left(\frac{1}{32}-\max_{j\in\{n_k+1,\dots,m_k+1\}}\frac{7+\log_2 j}{\ln^2(j+1)}\right)\\ &= \frac{1}{32}.\end{aligned}$$

Replacing g by $g/32$ gives us the desired conclusion. ∎

5.4 Remark: The proof of Theorem 5.3 can be simplified if we do not require g to be concave, but since concavity is often a convenient assumption in working with entropy convergence rates, we preferred to include the concavity requirement in the statement of the theorem.

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